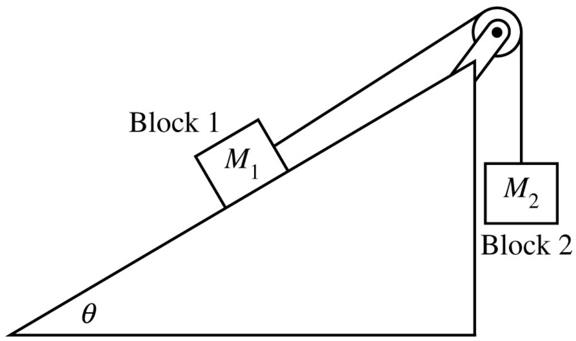
Scoring Guidelines

Question 1



1. Students set up a system of two blocks and an inclined plane, as shown in the figure. Block 1 of mass M_1 is on an surface that is inclined at an angle θ to the horizontal. The friction between block 1 and the surface is negligible. A string is attached to block 1, extends over an ideal pulley, and is then attached to block 2 of mass M_2.

   (a) In an initial setup, $M_1 = 3M$ and $M_2 = M$. Calculate the value of θ that would allow the system to remain in equilibrium.

   The original inclined plane is now replaced with one that has a rough surface. The coefficients of static and kinetic friction between block 1 and the surface are μ_s and μ_k, respectively. Block 1 is again chosen so that $M_1 = M$.

   (b) Derive an expression for the maximum value of M_2 that would allow this system to remain in equilibrium. Express your answer in terms of M, μ_s, μ_k, and physical constants, as appropriate.

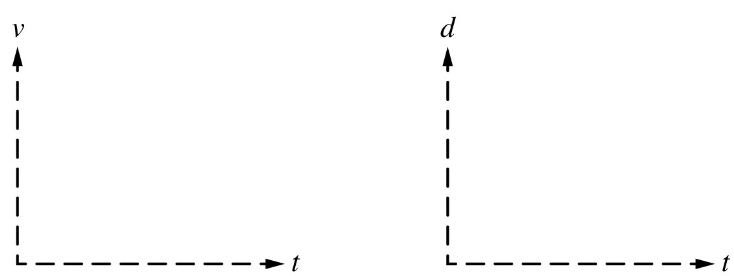
   Block 2 of mass M_2 is now chosen such that block 1 will accelerate up the inclined plane.

   (c)

   i. Derive an expression for the magnitude of the acceleration of block 1. Express your answer in terms of M_1, M_2, μ_s, μ_k, θ, and physical constants, as appropriate.

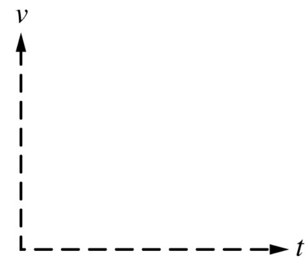
   ii. Derive an expression for the tension in the string. Express your answer in terms of M_1, M_2, μ_s, μ_k, θ, and physical constants, as appropriate.

   (d) On the axes below, sketch the speed v and distance d moved by block 1 up the inclined plane as functions of time.

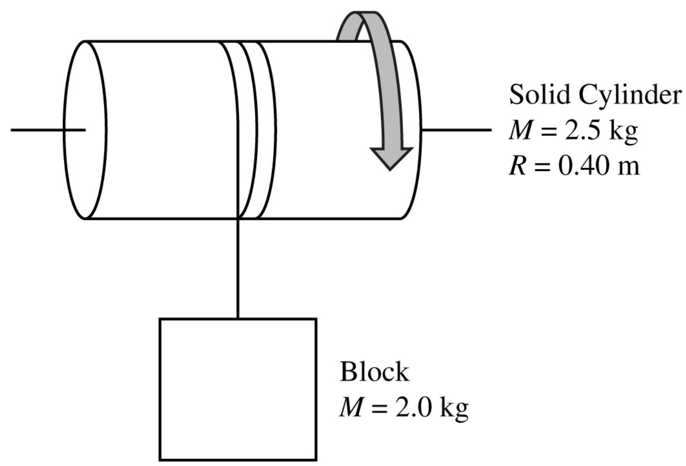


(e) During the experiments, students collect data that shows the acceleration of the blocks actually increases while the blocks are in motion.

i. On this axis below, sketch the speed v of block 1 as a function of t.



ii. Explain why the experiment may have produced an increasing acceleration instead of the predicted constant acceleration.



Scoring Guidelines for Question 1 — 15 points

Learning Objectives: CHA-1.C INT-1.B.a INT-1.C.e INT-3.A.c INT-3.B

(a) Calculate the value of θ that would allow the system to remain in equilibrium. — **1 point** 5.A

One point for a correct equation using Newton's second law on the two-block system in equilibrium.

$\sum F = M_1 g \sin\theta - M_2 g = (M_1 + M_2)a = 0$

One point for a correct substitution into the above equation. — **1 point** 6.B

$3Mg\sin\theta - Mg = 0$

$\theta = \sin^{-1}\left(\frac{1}{3}\right) = 19.5°$

Total for Part (a) — **2 points**

(b) Derive an expression for the maximum value of M_2 that would allow this system to remain in equilibrium. — **1 point** 5.A

One point for a correct equation using Newton's second law on the two-block system in equilibrium.

$\sum F = M_2 g - M_1 g \sin\theta - f = (M_1 + M_2)a = 0$

One point for a correct substitution for friction into the above equation. — **1 point** 5.D

$M_2 g = Mg\sin\theta + \mu_S F_N$

One point for a correct substitution for the normal force into the above equation. — **1 point** 5.D

$M_2 g = Mg\sin\theta + \mu_S Mg\cos\theta$

$M_2 = M(\sin\theta + \mu_S \cos\theta)$

Total for Part (b) — **3 points**

(c) **i.** Derive an expression for the magnitude of the acceleration of block 1. — **1 point** 5.A

One point for a correct substitution for friction into an equation using Newton's second law on the two-block system.

$\sum F = M_2 g - M_1 g \sin\theta - f = (M_1 + M_2)a$

$M_2 g - M_1 g\sin\theta - \mu_k F_N = (M_1 + M_2)a$

One point for a correct substitution for the normal force into the above equation. — **1 point** 5.D

$M_2 g - M_1 g\sin\theta - \mu_k M_1 g\cos\theta = (M_1 + M_2)a$

$$a = \frac{M_2 - M_1(\sin\theta - \mu_k \cos\theta)}{(M_1 + M_2)} g$$

ii. Derive an expression for the tension in the string. — **1 point** 5.A

One point for a correct expression of Newton's second law for block 2.

$\sum F = M_2 g - T = M_2 a$

One point for substituting the answer from part (c)(i) for the acceleration into the above equation. — **1 point** 5.D

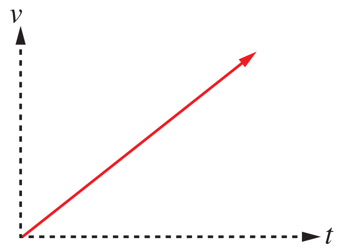
$T = M_2(g - a)$

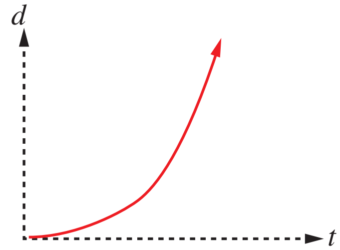
$$T = M_2\left(g - \frac{M_2 - M_1(\sin\theta - \mu_k\cos\theta)}{(M_1 + M_2)} g\right) = M_2 g\left(1 - \frac{M_2 - M_1(\sin\theta - \mu_k\cos\theta)}{(M_1 + M_2)}\right)$$

Total for Part (c) — **4 points**

(d) Sketch the speed v and distance d moved by block 1 up the inclined plane as functions of time. **1 point** 3.C

One point for indicating that both the speed and the distance increase with time.





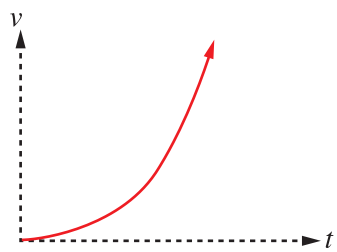
One point for a straight line with a positive slope for the v-t graph. **1 point** 3.C

One point for a concave up curve for the d-t graph. **1 point** 3.C

Total for Part (d) **3 points**

(e) **i.** On this axis provided, sketch the speed v of block 1 as a function of t. **1 point** 3.C

One point for a concave up curve for the v-t graph.



ii. Explain why the experiment may have produced an increasing acceleration instead of the predicted constant acceleration. **1 point** 7.A

One point for providing evidence to support the claim (The incline is smoother).

Example of acceptable evidence:

- The block's acceleration increases.
- The net force on the block increases.
- Greater friction indicates a rougher surface; less friction indicates a smoother surface.

One point for correct reasoning. **1 point** 7.D

Example of acceptable reasoning:

- The increase in the block's acceleration would indicate a smaller resistive force; thus, friction would be less which would be indicative of a smoother surface.

Example of acceptable explanation (claim, evidence, and reasoning):

- *The block's acceleration would increase if the top part of the incline is smoother than the bottom part. A smoother surface would result in a decrease in friction and an increase in the net force exerted on the block; thus, the block's acceleration would increase.*

Total for part (e) **3 points**

Total for Question 1 **15 points**

Question 2

2. A block of mass 2.0 kg is attached to a light string that is wrapped around a solid cylinder, as shown in the figure. The cylinder has a mass of $M = 2.5 \text{ kg}$ and a radius of $R = 0.40 \text{ m}$. The cylinder can rotate with negligible friction about a light rod through its central axis. The block-cylinder system is initially held at rest.

(a) Using integral calculus, show that the rotational inertia of the cylinder about its central axis is $\frac{1}{2}MR^2$.

(b) The block is released from rest and the string unwinds, causing the cylinder to rotate on the rod.

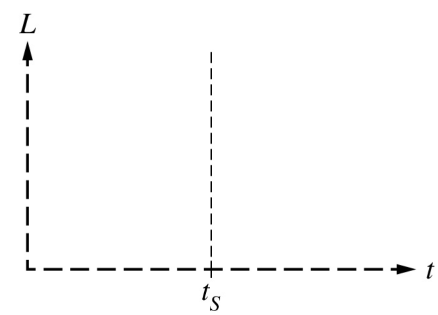
i. Calculate the linear acceleration of the block.

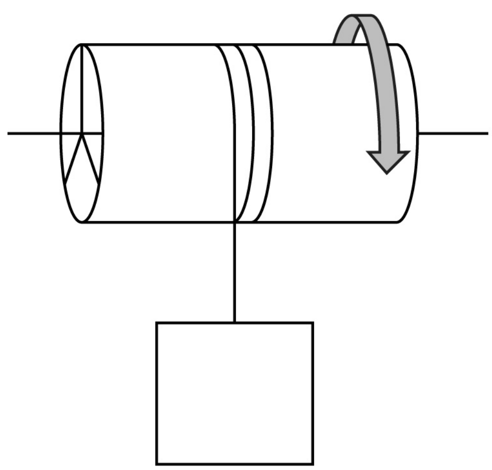
ii. Calculate the net torque exerted on the cylinder.

iii. Calculate the tension in the string.

At time t_S, the block reaches its lowest point as the string has completely unwound. The string then begins to rewind on the cylinder, and the mass is raised back upward.

(c) On the axis below, sketch the angular momentum L of the cylinder as a function of time t from the moment the mass is released to shortly after t_S.





Hollow Cylinder

The solid cylinder is replaced by a hollow cylinder with the same mass and radius. Lightweight spokes attach the hollow cylinder to a light rod through its central axis. The hollow cylinder can rotate around its central axis with negligible friction. The string is wound around the hollow cylinder so that the block is at the same initial position as before. The block is again released from rest. The time it takes for the string to completely unwind from the hollow cylinder is t_H.

(d) Is the time t_H greater than, less than, or equal to the time t_S ?

__________ Greater than __________ Less than __________ Equal to

Justify your answer.

Scoring Guidelines for Question 2 — 15 points

Learning Objectives: INT-1.C.e INT-3.B INT-6.D.e INT-7.A.b CON-5.A.b

(a) Using integral calculus, show that the rotational inertia of the cylinder about its central axis is $\frac{1}{2}MR^2$. — **1 point** 5.A

One point for using the integral form of the rotational inertia equation.

$$I = \int r^2 dm$$

$$V = \pi r^2 L = \frac{m}{\rho}$$

$$dm = 2\rho\pi rL dr$$

One point for a correct substitution for dm into the above equation. — **1 point** 5.D

$$I = \int r^2 (2\rho\pi rL dr) = 2\rho\pi L \int r^3 dr$$

One point for integrating with correct limits or constant of integration. — **1 point** 5.E

$$I = 2\rho\pi L \int_{r=0}^{r=R} r^3 dr = 2\left(\frac{M}{\pi R^2 L}\right)\pi L\left[\tfrac{1}{4}r^4\right]_{r=0}^{r=R} = \left(\frac{2M}{R^2}\right)\left(\frac{R^4}{4}\right) = \tfrac{1}{2}MR^2$$

Total for Part (a) — 3 points

(b) i. Calculate the linear acceleration of the block. — **1 point** 5.A

One point for a correctly substituting into the linear form of Newton's second law on the block.

$$\sum F = Mg - T = Ma$$

One point for a correctly substituting into the rotational form of Newton's second law on the cylinder. — **1 point** 5.A

$$\sum \tau = Fr_{\perp} = TR = I\alpha = \left(\tfrac{1}{2}MR^2\right)\left(\frac{a}{R}\right)$$

$$T = \tfrac{1}{2}Ma$$

One point for a correct expression of Newton's second law on the block-cylinder system. — **1 point** 6.C

$$Mg - \left(\tfrac{1}{2}Ma\right) = Ma$$

$$Mg = \tfrac{3}{2}Ma$$

$$a = \tfrac{2}{3}g = \tfrac{2}{3}\left(9.8\ \tfrac{\text{m}}{\text{s}^2}\right) = 6.5\ \tfrac{\text{m}}{\text{s}^2}$$

ii. Calculate the net torque exerted on the cylinder. — **1 point** 5.D

One point for correctly substituting the answer from part (b) into the rotational form of Newton's second law on the cylinder.

$$\sum \tau = \left(\tfrac{1}{2}MR^2\right)\left(\frac{a}{R}\right) = \left(\tfrac{1}{2}MR\right)\left(\tfrac{2}{3}g\right) = \tfrac{1}{3}MgR$$

One point for substitution into the above equation. — **1 point** 6.C

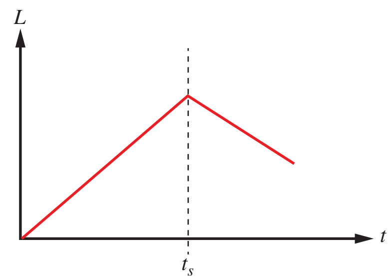
$$\tau = \left(\tfrac{1}{3}\right)(2.5\text{ kg})\left(9.8\ \tfrac{\text{m}}{\text{s}^2}\right)(0.40\text{ m}) = 3.27\ \text{N}\bullet\text{m}$$

iii. Calculate the tension in the string. — **1 point** 6.C

One point for substitution consistent with answer from (b)(i) into an equation to solve for tension.

$$T = \tfrac{1}{2}Ma = \left(\tfrac{1}{2}\right)(2.5\text{ kg})\left(6.5\ \tfrac{\text{m}}{\text{s}^2}\right) = 8.12\text{ N}$$

Total for Part (b) — 6 points

(c)	On the axis provided, sketch the angular momentum L of the cylinder as a function of time t from the moment the mass is released to shortly after t_S. One point for indicating that the linear momentum increases before t_S and decreases after t_S.  	**1 point** 3.C
	One point for a straight line with a positive slope for before t_S.	**1 point** 3.C
	One point for a straight line with a negative slope for after t_S.	**1 point** 3.C
	Total for Part (c)	**3 points**
(d)	Is the time t_H greater than, less than, or equal to the time t_S? One point for selecting "Greater than".	**1 point** 7.A
	One point for a correct justification that includes an indication that the rotational inertia is greater for the hollow cylinder than the solid sphere.	**1 point** 7.D
	One point for a correct justification that connects an increase in rotational inertia to both a decrease in acceleration and an increase of the time of fall. Example of acceptable justification: • *The hollow cylinder has more mass toward the outside of the cylinder; thus, it has a greater rotational inertia. Therefore, the acceleration decreases, and the time of fall increases.*	**1 point** 7.D
	Total for Part (d)	**3 points**
	Total for Question 2	**15 points**